Thermoelastic and Electromagnetic Damping Analysis

Usik Lee*
Stanford University, Stanford, California

The thermoelastic damping due to thermal currents and the electromagnetic damping due to electric conduction currents of vibrating solids are discussed. The effects of structural and geometrical constraints on damping loss factors are investigated. Also, optimum conditions for the maximum damping, which may be useful on the stage of system design, are investigated. It is found that damping loss factors are generally dependent upon structural and geometrical configurations. An analogy exists between thermoelastic damping and electromagnetic damping, showing Debye curves with Debye peaks. Standing transverse waves are likely to achieve larger damping than standing dilatational waves in the presence of a magnetic field. Electromagnetic damping in ferromagnetic material bodies is found to be considerable in high field. The influence of thermoelastic damping on aeroelastic stability of beam plates is investigated. This research strongly suggests that thermoelastic damping improves the aeroelastic stability of beam plates.

Nomenclature

a_p = Fourier coefficient
A = vector potential
C_H, C_u = electromagneto-elastic coupling vectors
C_k = amplification factor
c_v = constant-strain specific heat (unit mass)
D $\equiv Eh^3/12(1-\nu^2)$, plate flexural rigidity
e $\equiv \epsilon_{11}+\epsilon_{22}+\epsilon_{33}$
E = Young's modulus
$f(\alpha_3)$ = function defined in Eq. (4)
F, f = external forces
h = shell thickness
$\boldsymbol{h}$ $= \boldsymbol{H}-\boldsymbol{H}_0$, perturbation in magnetic field
$\boldsymbol{H}_0$ = constant reference magnetic field
i $\equiv \sqrt{-1}$, imaginary unit
k = thermal conductivity
K $\equiv Eh/(1-\nu^2)$
K_{11}, K_{22} = changes in curvatures of the middle surface
L = beam-plate length in x direction
L_1, L_2 = dimensions along α_1, α_2 coordinates
m_{ij} = Maxwell stress tensor, Eq. (47)
M = Mach number
M_k, M_m = quantities defined in Eqs. (10) and (57)
R_1, R_2 = radii of shell curvatures in α_1, α_2 coordinates
R = quantity defined in Eq. (49)
t = time coordinate
t_{ij} $\equiv \sigma_{ij}+m_{ij}$, total stress tensor
ΔT $\equiv T-T_0$, temperature disturbance
T_0 = constant reference absolute temperature
$\boldsymbol{u}$ = displacement field vector
U = strain energy
ΔU = energy dissipated
U_∞ = fluid velocity
w = deflection of shell in α_3 coordinate
W = normal mode (with subscripts)
α = velocity of longitudinal (P−) wave
$\alpha_1, \alpha_2, \alpha_3$ = orthogonal curvilinear coordinates
α_t = linear thermal coefficient of expansion
β = velocity of transverse (S−) wave
γ $\equiv \rho_\infty L/\rho h$, dimensionless mass ratio
λ_n = eigenvalue, Eq. (38a)
Γ_k = quantity defined in Eq. (14)
δ = logarithmic decrement (with subscripts)
δ = delta function
δ_{kl} = Kronecker delta
Δ $\equiv T_0\alpha^2 E/\rho c_v$, thermal relaxation strength
ϵ_1, ϵ_2 = electromagneto-elastic coupling constants
ϵ_{ij} = mechanical strain tensor
η = damping loss factor (with subscripts)
η_D $\equiv \Delta[\omega\tau/(1-\omega^2\tau^2)]$, Debye formula
η_t = quantity defined in Eq. (13)
θ = dimensionless constant ($\cong 1$)
μ_H = magnetic permeability
ν = Poisson's ratio
ν_H $\equiv 1/\sigma\mu_H$, magnetic viscosity
ξ = dimensionless quantity (with subscripts)
Π = factor defined in Eq. (23)
ρ = material density
ρ_∞ = fluid density
σ = electric conductivity
σ_{ij} = mechanical stress tensor
τ $\equiv \rho c_v h^2/\pi^2 k$, dimensionless time
ϕ = scalar potential
ψ = normal mode (with subscripts)
χ = dimensionless quantity (with subscripts)
ω = frequency
ω^* = characteristic frequency
$\Delta\omega$ $\equiv \omega_1-\omega_2$, half-power bandwidth
∇ = vector Nabla operator
$(\dot{\ })$ $\equiv \partial/\partial t$, time derivative

Subscripts

k = property of kth vibration mode
max = maximum value

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*Graduate Student, Department of Mechanical Engineering.

Introduction

MATERIAL damping arises from several physical sources and is therefore difficult to predict accurately. Nevertheless, reasonably accurate damping information is often required to design a system properly for dynamic loadings. There is considerable literature on both analytical and experimental aspects of the subject. Lazan[1] and Nowick and Berry[2] provide useful summaries of what was known up to their dates of publication. The author has found, however, relatively few fundamental theoretical studies of internal energy dissipation (or material damping).

The inherent dissipation in monolithic solids tends to be small compared to the damping furnished artificially by dashpots, constrained viscoelastic layers, interconnections, joints, and bearings. This is believed to explain why the role of material damping is frequently omitted or underplayed in the extensive literature on damping analysis and active control of Large Space Structures (LSS). Several authors (e.g., Gevarter,[3] Ashley[4]) have given some consideration to the possibly important role of material damping on the stabilization of structures. Reference 4 observed that a tiny amount of structural damping is useful for meeting the control system requirements of LSS like telescopes and antennas in space.

In order to analyze the internal energy dissipation of a given structure, one should take into account all possible damping mechanisms, depending upon the specific material. In practical cases, however, one or two mechanisms generally predominate, the others being comparatively negligible.

In 1938, Zener[5] predicted that thermoelastic damping (thermal damping) of monolithic crystalline solids is often much greater than the damping due to all other mechanisms. Experiments of Bennewitz and Rötger[6] confirmed that his predictions are accurate. Thermal damping is almost universal. But, under certain conditions with high electromagnetic (EM) fields, the electromagneto-elastic damping (simply electromagnetic damping) is of an even larger order of magnitude.

Thermal Damping Analysis

Background

Zener[5,7] was apparently the first to point out that the energy dissipation in vibrating metals must be sought in stress inhomogeneities, giving rise to temperature gradients and hence to local thermal currents, which increase the entropy. Biot[8] discussed irreversible thermodynamics in vibrating systems and applied a generalized coordinate method to the calculation of internal energy dissipation. Tasi and Herrmann[9,10] investigated a crystal plate by means of a variational principle. Chadwick,[11] in 1962, showed that his results from normal mode analysis agree with Zener's theory.[5] Later, Alblas[12] developed a general theory of energy dissipation in a three-dimensional finite body.

Basic Formulation

The two governing equations of the linearized coupled thermoelasto-dynamics are given by[13,14]

$$D\nabla_1^8 w + \left(\frac{E}{1-\nu}\right)\nabla_1^6 \int_{-h/2}^{h/2} \alpha_t \Delta T \alpha_3 \mathrm{d}\alpha_3 + Eh\nabla_k^4 w + \rho h \nabla_1^4 \ddot{w} = \nabla_1^4 F \tag{1}$$

$$\nabla^2(\alpha_t \Delta T) - \left(\frac{\rho c_v}{k}\right)\frac{\partial(\alpha_t \Delta T)}{\partial t} = \frac{T_0 \alpha_t^2 E}{k(1-2\nu)}\frac{\partial e}{\partial t} \tag{2}$$

where

$$\nabla_k^2(\cdot) = \frac{1}{A_1 A_2}\left\{\frac{\partial}{\partial\alpha_1}\left[\frac{1}{R_2}\frac{A_2}{A_1}\frac{\partial(\cdot)}{\partial\alpha_1}\right] + \frac{\partial}{\partial\alpha_2}\left[\frac{1}{R_1}\frac{A_1}{A_2}\frac{\partial(\cdot)}{\partial\alpha_2}\right]\right\}$$

A_1, A_2 are Lamé parameters and ∇_1^2 and ∇^2 are two- and three-dimensional Laplacian operators in orthogonal curvilinear coordinates $(\alpha_1, \alpha_2, \alpha_3)$, respectively. F is the external force and the other symbols are defined in the Nomenclature.

Equation (1) is the equation of transverse motion, which is believed sufficiently accurate to estimate quickly the effects of curvatures in relatively shallow shells. Equations for plates and beams are readily recovered by forcing the radii of shell curvatures, $R_1 = R_2 = \infty$ and Poisson's ratio $\nu = 0$. Because of this adaptability, plates and beams are easily recovered from the results obtained by solving Eqs. (1) and (2) for shallow shells. Equation (2) is the heat conduction equation, in which the influence of shell curvatures on thermal flux is neglected. One notes that the two governing equations include small coupling terms between elastomechanical and thermodynamic behaviors, which give rise to the damping of the vibration.

The general theory of shallow shells usually assumes, as above, that the normal stress σ_{33} is negligible along with shear strains ϵ_{13} and ϵ_{23}. Under this assumption, the reduced Hooke's law and the strain-displacement relations[13] give approximately the dilatational part of the displacement field (dilatation) in the form

$$e = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} \cong -\left[\frac{1-2\nu}{1-\nu}\right]\alpha_3 \nabla_1^2 w + \left[\frac{1+\nu}{1-\nu}\right]\alpha_t \Delta T \tag{3}$$

From the physics of the situation and the forms of Eqs. (2) and (3), a logical approximation to the elastomechanical coupling would seem to be

$$\alpha_t \Delta T(\alpha_1, \alpha_2, \alpha_3, t) \cong f(\alpha_3)\nabla_1^2 w(\alpha_1, \alpha_2, t) \tag{4}$$

which reduces Eqs. (1) and (2) to the forms:

$$D(1 + i\eta_t)\nabla_1^8 w + Eh\nabla_k^4 w + \rho h \nabla_1^4 \ddot{w} = \nabla_1^4 F \tag{5}$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\alpha_3^2} + \left[\frac{\nabla_1^4 w}{\nabla_1^2 w} - \left(\frac{\rho c_v \theta}{k}\right)\frac{\nabla_1^2 \dot{w}}{\nabla_1^2 w}\right] f + \left[\frac{\rho c_v \Delta}{k(1-\nu)}\frac{\nabla_1^2 \dot{w}}{\nabla_1^2 w}\right]\alpha_3 = 0 \tag{6}$$

where

$$\eta_t \equiv \text{Imag. part of } \left\{\frac{E}{D(1-\nu)}\int_{-h/2}^{h/2} f(\alpha_3)\alpha_3 \mathrm{d}\alpha_3\right\}$$

$$\Delta \equiv \frac{T_0 \alpha_t^2 E}{\rho c_v}, \qquad \theta \equiv 1 + \Delta\left[\frac{1+\nu}{(1-\nu)(1-2\nu)}\right] \cong 1 \tag{7}$$

One notes that only the imaginary part of the integration in Eq. (7) is taken because of its contribution to the damping of vibration. The modified equation of motion (5) now contains the complex plate flexural rigidity $D(1 + i\eta_t)$. θ is approximately equal to unity because always the thermal relaxation strength $\Delta \ll 1$.[7]

In order to investigate free vibration, one assumes harmonic motion in the form

$$w = \sum_{mn}^{\infty} C_{mn} W_{mn} e^{i\omega t} = \sum_{k}^{\infty} C_k W_k e^{i\omega t} \tag{8}$$

(Indices mn are replaced by k for convenience.) Here C_k is the amplification factor of the kth normal mode W_k, which satisfies the following equations:

$$D\nabla_1^8 W_k + Eh\nabla_k^4 W_k - \rho h \omega_k^2 \nabla_1^4 W_k = 0 \tag{9}$$

$$\int_A W_k W_l \mathrm{d}A = M_k \delta_{kl} \tag{10}$$

where $\mathrm{d}A = \mathrm{d}\alpha_1 \mathrm{d}\alpha_2$ is the plate area element and δ_{kl} is the Kronecker delta. It is convenient to expand $f(\alpha_3)$ in Fourier series, following the lead of Zener,[5]

$$f(\alpha_3) = \sum_{p=0}^{\infty} a_p \sin(2p+1)\frac{\pi\alpha_3}{h} \tag{11}$$

which satisfies insulated boundary conditions at the upper and lower shell surfaces. These are appropriate to the vacuum of space, and one assumes no energy loss due to heat convection or radiation. Substituting Eqs. (8) and (11) into Eq. (6) and using the orthogonality property of Fourier series, one may solve for the coefficient a_p of Eq. (11) to find that $a_p \ll a_0$ for $p>1$. Therefore, a one-term approximation is acceptable with an error typically $<2\%$,

$$f(\alpha_3) \cong [f_R + if_I]\sin(\pi\alpha_3/h) \tag{12}$$

where

$$f_R = \frac{1}{\pi^2}\frac{4h}{1-\nu}\Delta\frac{\omega^2\tau^2}{1+\omega^2\tau^2}$$

$$f_I = \frac{1}{\pi^2}\frac{4h}{1-\nu}\Delta\frac{\omega\tau}{1+\omega^2\tau^2}$$

Here we used the approximations $\theta \cong 1$ and $\mathcal{O}[(h/L)^2] \cong 0$ for the shallow shells. τ is the characteristic time, which is controlled by the choice of material, specimen shape and size, and defined by $\tau \equiv (\rho c_v h^2/\pi^2 k)$. Substituting Eq. (12) into Eq. (7) gives

$$\eta_t \cong \frac{96}{\pi^4}\left(\frac{1+\nu}{1-\nu}\right)\left[\Delta\frac{\omega\tau}{1+\omega^2\tau^2}\right] \tag{13}$$

Here the square-bracketed portion of Eq. (13) is called the Debye formula, η_D.

Free Vibration

Consider free vibration with forcing $F=0$, and assume harmonic motion, Eq. (8). Then, using the orthogonality property of Eq. (10), the equation of motion can be reduced to the form

$$\rho h(\omega^2 - \omega_k^2) + i\eta_t(\Gamma_k - \rho h\omega_k^2) = 0 \tag{14}$$

where

$$\Gamma_k = \Gamma_{mn} = Eh(\nabla_k^4 W_k / \nabla_1^4 W_k)$$

Because of small η_t, Γ_k is treated as a constant without causing significant error. In fact, Γ_k is found to be constant for the simply supported structures with harmonic motions considered in this study.

One measure of free oscillation decay is the logarithmic decrement, which yields the modal damping loss factor, as follows:

$$\frac{\delta_k}{\pi} \cong \frac{96}{\pi^4}\left(\frac{1+\nu}{1-\nu}\right)\Delta\frac{\omega_k\tau}{1+\omega_k^2\tau^2}\left[\frac{\bar{\omega}_k}{\omega_k}\right]^2 \tag{15}$$

where

$$\bar{\omega}_k^2 = \omega_k^2 - (\Gamma_k/\rho h)$$

Forced Vibration

Consider the vibration forced by a concentrated load P_0 acting at point $(\bar{\alpha}_1, \bar{\alpha}_2)$. In terms of modal modes governed by Eq. (9), Eq. (5) can be written

$$\rho h\sum_k^{\infty}(\omega_k^2 - \omega^2)C_k W_k - i\eta_t\sum_k^{\infty}(\Gamma_k - \rho h\omega_k^2)C_k W_k$$

$$= P_0\delta(\alpha_1 - \bar{\alpha}_1)\delta(\alpha_2 - \bar{\alpha}_2) \tag{16}$$

Using again the orthogonality property of Eq. (10), one finds the amplification factor C_k as follows:

$$C_k = \frac{P_0 W_k(\bar{\alpha}_1, \bar{\alpha}_2)}{\rho h M_k[(\omega_k^2 - \omega^2) + i\eta_t(\omega_k^2 - \Gamma_k/\rho h)]} \tag{17}$$

As an estimate of damping, the half-power bandwidth $\Delta\omega = \omega_1 - \omega_2$ is readily obtained from the amplification factor C_k. Then one measure of damping for the kth mode of vibration is simply

$$\frac{\Delta\omega}{\omega_k} \cong \frac{96}{\pi^4}\left(\frac{1+\nu}{1-\nu}\right)\Delta\frac{\omega_k\tau}{1+\omega_k^2\tau^2}\left[\frac{\bar{\omega}_k}{\omega_k}\right]^2 \tag{18}$$

which proved identical to the logarithmic decrement, Eq. (15).

Another classical measure of damping is the loss factor η, defined as the ratio of energy dissipated in unit volume per radian of oscillation to the maximum strain energy per unit volume, that is,

$$\eta = \Delta U / 2\pi U_{\max} \tag{19}$$

Here

$$U \cong \frac{1}{2}\int_A \int_{-h/2}^{h/2} [\sigma_{11}\epsilon_{11} + \sigma_{22}\epsilon_{22}]\,d\alpha_3 dA \tag{20}$$

$$\Delta U \cong \int_A \int_{-h/2}^{h/2} \int_0^{2\pi} [\sigma_{11}\dot{\epsilon}_{11} + \sigma_{22}\dot{\epsilon}_{22}]\,d\omega t\, d\alpha_3 dA \tag{21}$$

In Eqs. (20) and (21), approximations have been made consistent with the foregoing derivations. Space limitations prevent reproducing detail of a consistent analysis, which leads to the expression

$$\eta = \frac{96}{\pi^4}\left(\frac{1+\nu}{1-\nu}\right)\Delta\frac{\omega\tau}{1+\omega^2\tau^2}\Pi \tag{22}$$

with

$$\Pi = \int_A [K_{11} + K_{22}]^2 dA \Big/ \int_A [K_{11}^2 + 2\nu K_{11}K_{22} + K_{22}^2]\,dA \tag{23}$$

where K_{11} and K_{22} are the changes in curvatures of the middle surface.[13] In any practical examples, $\Pi > 1$ since $\nu < 0.5$. However, one notes that $\Pi = 1$ for the structures vibrating one-dimensionally like beam plates. For the simply supported structures considered in this paper, Π has the general form

$$\Pi = \left\{\sum_k^{\infty} \bar{C}_k^2\left[\left(\frac{m\pi}{L_1}\right)^2 + N^2\left(\frac{n\pi}{L_2}\right)^2\right]^2\right\}$$

$$\div\left\{\sum_k^{\infty}\bar{C}_k^2\left[\left(\frac{m\pi}{L_1}\right)^4 + 2\nu N^2\left(\frac{m\pi}{L_1}\right)^2\left(\frac{n\pi}{L_2}\right)^2\right.\right.$$

$$\left.\left. + N^4\left(\frac{n\pi}{L_2}\right)^4\right]\right\} \tag{24}$$

where $\bar{C}_k$ is the magnitude of C_k, Eq. (17). L_1 and L_2 are the full dimensions of a structure along the coordinates α_1 and α_2, respectively. Note that $N=1$ for the rectangular flat plates and curved panels and that $N=2$ for the cylindrical shells and barrel-shaped shells.[13]

Investigation of amplification factor C_k shows that the kth normal mode predominates when the circular frequency ω is near the kth natural frequency ω_k. Then one can use the approximation

$$\eta \cong \frac{96}{\pi^4}\left(\frac{1+\nu}{1-\nu}\right)\Delta\frac{\omega\tau}{1+\omega^2\tau^2}\Pi_k \qquad (\omega \simeq \omega_k) \tag{25}$$

where the Π_k (or Π_{mn}) for simply supported structures is given by

$$\Pi_{mn} = \left[\left(\frac{m\pi}{L_1}\right)^2 + N^2\left(\frac{n\pi}{L_2}\right)^2\right]^2 \div \left[\left(\frac{m\pi}{L_1}\right)^4 + 2\nu N^2\left(\frac{m\pi}{L_1}\right)^2\left(\frac{n\pi}{L_2}\right)^2 + N^4\left(\frac{n\pi}{L_2}\right)^4\right] \tag{26}$$

The kth modal loss factors for simply supported structures, without further approximation, are readily obtained from Eq. (22) in the form

$$\eta_k = \frac{96}{\pi^4}\left(\frac{1+\nu}{1-\nu}\right)\frac{\omega_k\tau}{1+\omega_k^2\tau^2}\Pi_k \tag{27}$$

Discussion

Equations (15), (18), and (27) provide measures of modal damping at frequencies near natural frequencies. For structures vibrating one-dimensionally, such as beams and beam plates, these equations give exactly identical results as in the case of a mass-spring-dashpot system. In general, however, there is no unique expression suitable as a measure of damping, even at a natural frequency. One may therefore ask which measure of damping is the most meaningful and accurate. The author has concluded that this question has no definitive answer. As Jones[15] observed, this ambiguity is really not a serious problem. When comparing different materials and configurations, one must simply employ consistent, clearly defined measures. As far as small damping is concerned, every measure must provide the same useful information. For this reason, the author has adopted the loss factor η as a vehicle for further investigations.

How to maximize the loss factor seems to be the most interesting issue for damping analysis. Maximization of damping is not a simple matter because of the complicated characteristics of vibration problems. Figures 1-3 have been calculated to illustrate factors Π and loss factors η for simply supported structures with the same surface areas (i.e., $L_1 = L_2 = 2m$). As a preliminary, Fig. 1 and the study of C_k and Eq. (17) demonstrate that factor Π is nearly independent of circular frequency and structural thickness. These factors are clearly important for the part of η_D of Eq. (13). Figure 2 shows that the loss factor increases at very low frequencies and decreases at high frequencies as the thickness increases. It also demonstrates that the loss factor is almost proportional to the reference absolute temperature.

From earlier developments for a given material and structure, η_D and Π can be represented by

$$\eta_D \cong \eta_D(\omega, h), \qquad \Pi \cong \Pi(L_1, L_2, m, n) \tag{28}$$

Then it is obvious that η_D has its maximum value at frequency $\omega \cong 1/\tau$, which is called the Debye peak. The thickness for maximum η_D is readily obtained from

$$h \cong \pi[k/\rho c_v \omega]^{1/2} \tag{29}$$

Since damping plays its most important role at frequencies near natural frequencies, it is valuable to maximize the modal factor Π_{mn} of Eq. (26). An optimal combination of geometry and mode of vibration for the maximum value of Π_{mn} is found to be

$$L_2/L_1 \cong N(n/m) \tag{30}$$

Equations (29) and (30) will be useful for designers who wish to maximize the damping of vibration of the sort considered here.

Consider the modal loss factors of the plates and shells which are made of the same material. At a natural frequency, when the same dimensions (L_1, L_2) and modes (m,n) are selected for the plate and shell, it follows that

$$\frac{\eta_s}{\eta_p} = \frac{\omega_s}{\omega_p}\,\frac{1+\omega_p^2\tau}{1+\omega_s^2\tau} \tag{31}$$

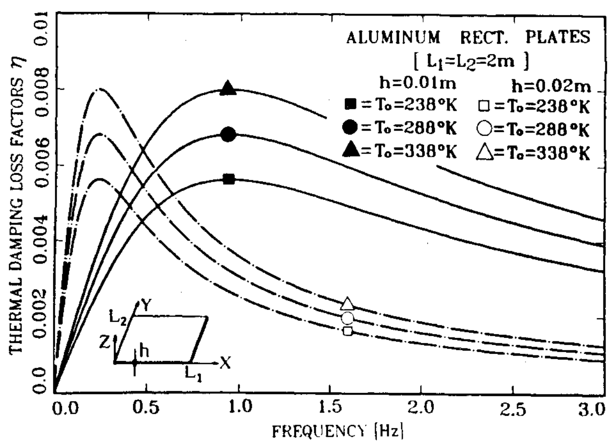

Fig. 2 Temperature and thickness dependence of loss factor η.

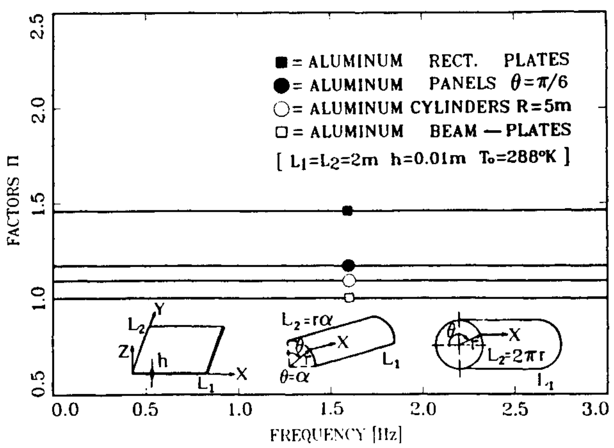

Fig. 1 Frequency dependence of factor Π.

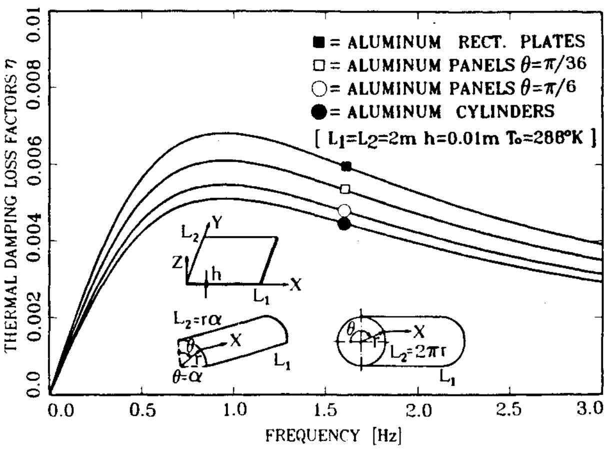

Fig. 3 Curvature dependence of loss factor η.

Since $\omega_p \omega_s > 1/\tau$ in general, the modal loss factor of a plate tends to be larger than that of a shell. Figures 1 and 3 show that plates do indeed have the largest damping, followed by panels, cylinders, beam plates, and simple beams. The damping of a panel gets closer to that of a plate as it gets flatter. Also the damping of a curved panel gets closer to that of a cylinder as it approaches the shape of the cylinder (see Fig. 3). It is also found that the damping of a barrel-shaped shell is larger than that of a cylinder, but again the former gets closer to the latter as the radius of barrel curvature increases. Figure 4 has been calculated to illustrate the modal damping loss factors for simply supported and partially clamped plates at the first five fundamental frequencies. Simply supported plates achieve higher damping than partially clamped plates.

Without observation of what is really happening inside the material, it seems to be very difficult to clarify the foregoing results with a reasonable physical interpretation. From the heat conduction equation (2), however, one can conclude that a structure will experience higher damping when the rate of dilatation gets larger. When the geometry and boundary conditions for a particular structure are likely to increase the rate of dilatation, the structure will achieve great damping. Constraints on a structure seem to prevent increasing the rate of dilatation with increasing natural frequency. Curved and clamped structures have more constraints than flat and simply supported structures.

Electromagnetic Damping Analysis

Background

Electromagneto-solid mechanics deals with the effects of an EM field on the elastic deformations of a solid body. In order to avoid an excessively long bibliography, the reader is referred to Refs. 16-18.

There are two main reasons, in the author's judgment, why until recently there existed relatively few applications of magneto-elasticity. First, EM effects were not a significant industrial problem, at least until the appearance of very powerful magnets. Second, continuum mechanics cannot avoid resorting to complicated formulations for predicting the internal forces in a magnetized body.

On the other hand, industrial activity could not avoid dealing with instruments working in a strong EM field. In 1963 Alers and Fleury[19] recognized from experiments that the influence of magneto-elastic interactions is considerable in many such situations. Moon[20] and Paria[21] discussed many practical applications of magneto-solid mechanics.

In extensive literature on the subject, very few studies of internal energy dissipation due to an EM field are known. Zener[5] is believed to be the first to predict energy dissipation by eddy currents in vibrating ferromagnetic metals. Chadwick[22] investigated the effect of a static magnetic field on wave motion in a conducting solid. Subsequently, Smith and Herrmann[23] discussed the effect of a type of magnetic damping upon the stability of some circulatory elastic systems. Nayfeh and Nasser[24] examined the influence of small thermoelastic and magneto-elastic couplings on the propagation of plane electromagnetic-elastic waves through an infinite isotropic medium.

In this paper, the author investigates the electromagnetic damping due to electric conduction currents within a homogeneous, isotropic, and electromagnetically linear conducting elastic body. The underlying theory is linear magneto-elasticity.[21,25]

Basic Formulation

A complete set of equations of the linear magneto-elasticity is given by

$$\alpha^2 \nabla \nabla \cdot \boldsymbol{u} - \beta^2 \nabla \times (\nabla \times \boldsymbol{u}) + (\mu_H/\rho)(\nabla \times \boldsymbol{h}) \times \boldsymbol{H}_0 + \boldsymbol{f} = \ddot{\boldsymbol{u}} \tag{32a}$$

$$\nabla^2 \boldsymbol{h} - (1/\nu_H)\dot{\boldsymbol{h}} = -(1/\nu_H) \nabla \times (\dot{\boldsymbol{u}} \times \boldsymbol{H}_0) \tag{32b}$$

where $\boldsymbol{u}$ is the displacement field and $\boldsymbol{h}$ is a small perturbation in a constant reference magnetic field $\boldsymbol{H}_0$. $\nu_H = 1/\sigma\mu_H$ is the magnetic viscosity, where σ is electric conductivity and μ_H is the magnetic permeability. Application of vector identities and properties of Maxwell's equations in Eqs. (32) yields

$$\alpha^2 \nabla \nabla \cdot \boldsymbol{u} - \beta^2 \nabla \times (\nabla \times \boldsymbol{u}) + \nabla \times (\boldsymbol{h} \times \boldsymbol{C}_H) - \nabla (\boldsymbol{h} \cdot \boldsymbol{C}_H) + \boldsymbol{f} = \ddot{\boldsymbol{u}} \tag{33a}$$

$$\nabla^2 \boldsymbol{h} - (1/\nu_H)\dot{\boldsymbol{h}} - \boldsymbol{C}_u (\nabla \cdot \dot{\boldsymbol{u}}) - \boldsymbol{C}_u \times (\nabla \times \dot{\boldsymbol{u}}) + \nabla (\dot{\boldsymbol{u}} \cdot \boldsymbol{C}_u) = 0 \tag{33b}$$

where $\boldsymbol{C}_H$ and $\boldsymbol{C}_u$ are the electromagneto-elastic coupling vectors defined by

$$\boldsymbol{C}_H \equiv \mu_H \boldsymbol{H}_0/\rho \qquad \boldsymbol{C}_u \equiv \boldsymbol{H}_0/\nu_H$$

Parabolic Eq. (33b) contains three types of coupling terms: dilatational, rotational, and gradient parts. In the case of the heat conduction equation, one sees by comparison that only a dilatational part exists.

To simplify the analysis without losing generality, one assumes that the gradient part can be rendered negligible by applying a properly oriented magnetic field. That is,

$$\nabla (\dot{\boldsymbol{u}} \cdot \boldsymbol{C}_u) = 0 \tag{34}$$

With the assumption, Eq. (34), one introduces the Helmholtz theorem[26]

$$\boldsymbol{u} = \nabla \phi + \nabla \times \boldsymbol{A}, \qquad \nabla \cdot \boldsymbol{A} = 0 \tag{35}$$

Assuming $\boldsymbol{f} = \nabla F$ and inserting Eq. (35) into Eqs. (33), one finds the equations as follows:

$$\alpha^2 \nabla^2 \phi - \boldsymbol{h} \cdot \boldsymbol{C}_H + F = \ddot{\phi} \tag{36a}$$

$$\beta^2 \nabla^2 \boldsymbol{A} + \boldsymbol{h} \times \boldsymbol{C}_H = \ddot{\boldsymbol{A}} \tag{36b}$$

$$\nabla^2 \boldsymbol{h} - (1/\nu_H)\dot{\boldsymbol{h}} + \boldsymbol{C}_u \times \nabla^2 \dot{\boldsymbol{A}} - \boldsymbol{C}_u \nabla^2 \dot{\phi} = 0 \tag{36c}$$

These three equations govern longitudinal (P−), transverse (S−), and EM waves. P and S waves are coupled to the EM wave.

Modal analysis is useful for the solution to Eqs. (36) in the form

$$\phi(\boldsymbol{r},t) = \sum_n^{\infty} \phi_n^0 \psi_n(\boldsymbol{r}) e^{i\omega t} \tag{37a}$$

$$\boldsymbol{h}(\boldsymbol{r},t) = \sum_n^{\infty} [\psi_n] \boldsymbol{h}_n^0 e^{i\omega t} \tag{37b}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \sum_n^{\infty} [\psi_n] \boldsymbol{A}_n^0 e^{i\omega t} \tag{37c}$$

with the harmonic excitation given by

$$F(\boldsymbol{r},t) = \sum_n^{\infty} F_n \psi_n(\boldsymbol{r}) e^{i\omega t} \tag{37d}$$

In Eqs. (37), $[\psi_n]$ shows the diagonal matrix with its elements ψ_n. Here $\boldsymbol{r}$ is a collective symbol for the coordinates x,y,z, and $\phi_n^0, \boldsymbol{h}_n^0, \boldsymbol{A}_n^0$ are constant magnification factors. Eigenfunctions ψ_n and corresponding eigenvalues λ_n are determined by

$$\nabla^2\psi_n + \lambda_n^2\psi_n = 0 \quad \text{on } V \tag{38a}$$

$$\nabla\psi_n \cdot \boldsymbol{n} = 0 \quad \text{on } S \tag{38b}$$

$$\int_V \psi_m\psi_n \mathrm{d}V = \delta_{mn} \tag{38c}$$

where $\boldsymbol{n}$ is the unit outward vector normal to surface S. Even though the boundary conditions (38b) do not embrace all possible realistic conditions at bounding surfaces,[25] the corresponding solutions should provide an acceptable basis for the analytical study of electromagnetic damping. Introducing assumed solutions into Eqs. (36), one may obtain

$$(\alpha^2\lambda_n^2 - \omega^2)\phi_n^0 + \boldsymbol{h}_n^0 \cdot \boldsymbol{C}_H = F_n \tag{39a}$$

$$(\beta^2\lambda_n^2 - \omega^2)\boldsymbol{A}_n^0 - \boldsymbol{h}_n^0 \times \boldsymbol{C}_H = 0 \tag{39b}$$

$$[\lambda_n^2 + (i\omega/\nu_H)]\boldsymbol{h}_n^0 + i\omega\lambda_n^2\boldsymbol{C}_u \times \boldsymbol{A}_n^0 - i\omega\lambda_n^2\boldsymbol{C}_u\phi_n^0 = 0 \tag{39c}$$

Here F_n is usually prespecified. When a constant magnetic field is applied in such a way that $\boldsymbol{C}_H = (C_{H1}, C_{H2}, 0)$, then Eqs. (39) are reduced to a 2×2 matrix equation with unknown variables $\boldsymbol{h}_n^0 = (h_{n1}, h_{n2}, 0)$.

Free Vibration

Put $F_n = 0$ in order to study free vibration at a natural frequency ω_n. Then Eqs. (39) yield two algebraic equations, as follows:

$$\chi_{n1}^3 - i\xi_{n1}^2\chi_{n1}^2 - \xi_{n1}^2(1+\epsilon_1)\chi_{n1} + i\xi_{n1}^4 = 0 \tag{40a}$$

$$\chi_{n2}^3 - i\xi_{n2}^2\chi_{n2}^2 - \xi_{n2}^2(1+\epsilon_2)\chi_{n2} + i\xi_{n2}^4 = 0 \tag{40b}$$

Here ϵ_1 and ϵ_2 are electromagneto-elastic coupling constants with standing S and P waves, respectively. The new notations are defined as follows:

$$\left(\frac{\beta^2}{\nu_H}, \frac{\alpha^2}{\nu_H}\right) \equiv (\omega_1^*, \omega_2^*) \qquad \left(\frac{\omega_n}{\omega_1^*}, \frac{\omega_n}{\omega_2^*}\right) \equiv (\chi_{n1}, \chi_{n2})$$

$$\left(\frac{\nu_H\lambda_n^2}{\omega_1^*}, \frac{\nu_H\lambda_n^2}{\omega_2^*}\right) \equiv (\xi_{n1}^2, \xi_{n2}^2) \qquad \left(\frac{\rho\sigma C_H^2}{\omega_1^*}, \frac{\rho\sigma C_H^2}{\omega_2^*}\right) \equiv (\epsilon_1, \epsilon_2)$$

where ω_1^*, ω_2^* are characteristic frequencies and the others are nondimensional quantities. When $\epsilon_1 = \epsilon_2 = 0$, the algebraic equations for uncoupled elastic S and P waves are recovered.

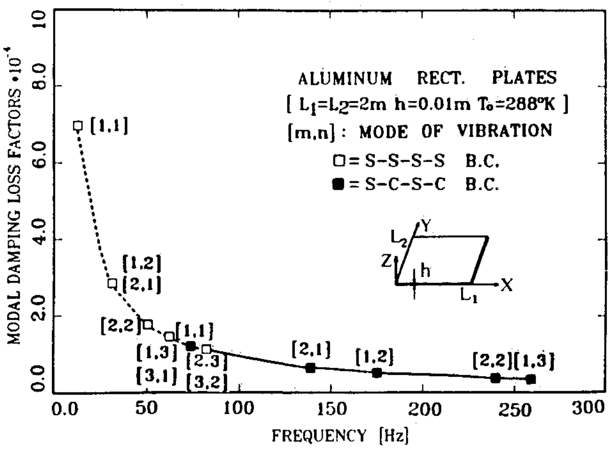

Fig. 4 Boundary condition dependence of modal loss factor η_k.

It is interesting to find that each of the algebraic equations (40) has a form identical to that for the thermal damping of an elastic body, solved by Chadwick.[11] This implies that electromagnetic damping is analogous to thermal damping. With the help of Chadwick's solutions obtained for the same type of algebraic equation, the modal damping loss factor can be obtained from

$$\frac{\delta_{ni}}{\pi} \cong \epsilon_i \frac{\xi_{ni}}{1+\xi_{ni}^2} \qquad (i = 1,2) \tag{41}$$

where higher terms $\mathcal{O}(\epsilon_i^2)$ are neglected. Since $\epsilon_1 > \epsilon_2$, standing S waves are seen to have larger damping than standing P waves.

As an example, consider an infinite plate with uniform thickness h. To obtain maximum dampings for the S and P waves, the thicknesses can be shown to be

$$h_1 = (2n+1)[\nu_H\pi/\beta] \quad \text{for the S wave} \tag{42a}$$

$$h_2 = (2n+1)[\nu_H\pi/\alpha] \quad \text{for the P wave} \tag{42b}$$

Therefore the standing S wave requires larger thickness than the standing P wave to achieve peak damping.

Forced Vibration

It is convenient to introduce a new notation, defined by $(\omega/\omega_2^*) \equiv \chi_2$. For the forced vibration, Eqs. (39) can readily be solved for ϕ_n^0, $\boldsymbol{A}_n^0$, and $\boldsymbol{h}_n^0$. Without reproducing complicated calculations, one finds the real parts of the solutions as follows:

$$\boldsymbol{u} = \sum_n^\infty [\phi_n^0 \cos\omega t]\nabla\psi_n \tag{43}$$

$$\boldsymbol{h} = \sum_n^\infty [\boldsymbol{h}_{nR}^0 \cos\omega t - \boldsymbol{h}_{nI}^0 \sin\omega t]\psi_n \tag{44}$$

where

$$\phi_n^0 \cong -\frac{F_k}{\omega_2^{*2}} \frac{\xi_{n2}^4 + \chi_2^2}{(\chi_2^4 + \xi_{n2}^2)(\chi_2^2 - \xi_{n2}^2) - \epsilon_2\xi_{n2}^2\chi_2^2}$$

$$\boldsymbol{h}_{nR}^0 \cong \frac{F_n \boldsymbol{C}_H}{C_H^2} \frac{\epsilon_2\chi_2^2\xi_{n2}^2}{(\xi_{n2}^4 + \chi_2^2)(\chi_2^2 - \xi_{n2}^2) - \epsilon_2\xi_{n2}^2\chi_2^2}$$

$$\boldsymbol{h}_{nI}^0 \cong \frac{F_n \boldsymbol{C}_H}{C_H^2} \frac{\epsilon_2\chi_2\xi_{n2}^4}{(\xi_{n2}^4 + \chi_2^2)(\chi_2^2 - \xi_{n2}^2) - \epsilon_2\xi_{n2}^2\chi_2^2}$$

The result shows that only the dilatation contributes to the damping when one assumes $\boldsymbol{f} = \nabla F$.

Equation (19) will be adopted as a measure of electromagnetic damping in forced vibration. In the present case, the energies U and ΔU are given in the form

$$U = \frac{1}{2}\int_V t_{ij}\epsilon_{ij}\mathrm{d}V \tag{45}$$

$$\Delta U = \int_V \int_0^{2\pi} m_{ij}\dot{\epsilon}_{ij}\mathrm{d}\omega t \mathrm{d}V \tag{46}$$

Note that t_{ij} is the total stress, which is the sum of mechanical stress and linearized Maxwell stress,

$$t_{ij} = \sigma_{ij} + m_{ij} \tag{47}$$

where

$$\sigma_{ij} = \frac{E}{(1+\nu)(1-2\nu)}[\nu e\delta_{ij} + (1-2\nu)\epsilon_{ij}]$$

$$m_{ij} = \mu_H[(H_{0i}h_j + H_{0j}h_i) - \boldsymbol{H}_0 \cdot \boldsymbol{h}\delta_{ij}]$$

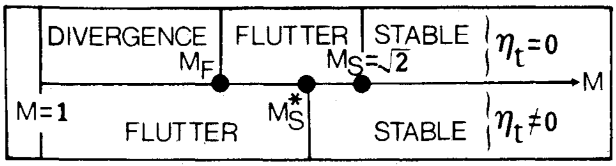

Fig. 5 Instability of single mode ($N=1$).

It is obvious that the mechanical stress does not contribute to the energy dissipation. Thus, only the Maxwell stress is considered in Eq. (46). It can be shown, by using the assumption (34), that the round bracketed part of m_{ij} also does not contribute to the energy dissipation. A substantial series of calculations, with application of the orthogonality property (38c), leads to

$$\eta \cong \epsilon_2 R \sum_n^{\infty} P_n \frac{\chi_2 \xi_{n2}^2}{\xi_{n2}^4 + \chi_2^2} \tag{48}$$

where

$$R \cong \int_V e_{\max}^2 \mathrm{d}V \Bigg/ \int_V \left[e^2 + \left(\frac{1-2\nu}{1-\nu}\right)(\epsilon_{ij}^2 - e^2)\right]_{\max} \mathrm{d}V \tag{49}$$

In Eq. (48), P_n satisfies the Parseval properties.[11] The perturbation of EM field generated in a solid body is likely to be so small that its contribution to R is neglected. Equation (48) is exactly analogous to the key results of Zener[5] and Chadwick,[11] which are for thermoelastic damping. Zener worked out a number of simple cases in which he estimated the values of P_n. It is found that $P_0 \cong 1$ and $P_n \ll 1$ for $n>1$. Equation (48) can then be approximated by

$$\eta \cong \epsilon_2 R [\chi_2 \xi_{02}^2 / (\chi_2^2 + \xi_{02}^4)] \tag{50}$$

which has the maximum value $\epsilon_2 R/2$ when $\chi_2 = \xi_{02}^2$. The modal loss factor at a natural frequency $\chi_2 \cong \xi_{n2}$ is found to be

$$\eta_n \cong \epsilon_2 R [\xi_{n2}/(1+\xi_{n2}^2)] \tag{51}$$

This result agrees with δ_{n2}/π apart from factor R (≈ 1). η_n has the maximum value $\epsilon_2 R/2$ when $\xi_{n2}=1$.

Modal damping loss factor, defined by the half-power bandwidth, is found from the magnification factor ϕ_n^0 of Eq. (43) in the form

$$\frac{\Delta\omega}{\omega_n} = \frac{\Delta\chi_2}{\chi_{n2}} \cong \epsilon_2 \frac{\xi_{n2}}{1+\xi_{n2}^2} \tag{52}$$

which is exactly the same as Eq. (41) with index $i=2$ and Eq. (51) with $R \cong 1$.

The general expression for electromagnetic damping shows that it takes the form of the Debye formula and that it is proportional to the square of the magnitude of the reference EM field. Therefore the contribution of electromagnetic damping to total material damping will be considerable in high fields.

Thermal Damping Effects on the Aeroelastic Stability of Beam Plates

Analysis

Consider a simply supported or clamped beam plate vibrating one-dimensionally. Dowell's equation of aeroelastic equilibrium[27] is modified into the form

$$D(1+i\eta_t)\frac{\partial^4 w}{\partial x^4} + \rho h \ddot{w} + \Delta_p^M + \Delta_p^E = 0 \tag{53}$$

This now includes a thermal damping source η_t. Familiar aerodynamic theory will be directly applied as given by

$$\Delta_p^M = \frac{\rho_\infty U_\infty^2}{(M^2-1)^{1/2}} \left[\frac{\partial w}{\partial x} + \frac{M^2-2}{M^2-1} \frac{1}{U_\infty} \frac{\partial w}{\partial t} \right] \tag{54}$$

which is the unsteady aerodynamic pressure due to plate motion in supersonic flow ($M>1$) and low frequency. Δp^E in Eq. (53) is the external aerodynamic pressure independent of plate motion. Even though aerodynamic damping, the second term of Eq. (54), changes sign from negative to positive as M increases above $\sqrt{2}$, thermal damping η_t is always positive. For thermal damping, the physics seems to be true because the feedback of temperature fluctuation during vibration always gives rise to energy dissipation.

Application of modal analysis and the assumption of harmonic motion lead to a characteristic equation in matrix form,

$$|M_n[\omega^2 - (1+i\eta_t)\omega_n^2]\delta_{mn} - \rho_\infty U_\infty^2 Q_{mn}^M| = 0 \tag{55}$$

Here the following relations have been used:

$$D\frac{\partial^4 W_n}{\partial x^4} - \rho h \omega_n^2 W_n = 0 \tag{56}$$

$$\int_0^L W_m W_n \mathrm{d}x = M_m \delta_{mn} \tag{57}$$

$$Q_{mn}^M = \frac{1}{\rho h (M^2-1)^{1/2}} \frac{M^2-2}{M^2-1} \frac{i\omega}{U_\infty} M_n \quad \text{for } m=n$$
$$= \frac{1}{\rho h (M^2-1)^{1/2}} \int_0^L \frac{\mathrm{d}W_m}{\mathrm{d}x} W_n \mathrm{d}x \quad \text{for } m \neq n \tag{58}$$

where L is the plate length in the streamwise x direction. One may replace η_t with its maximum value, independent of frequency, to simplify the analysis without compromising the main objective of the study.

Case I: Single Mode ($N=1$)

For the nth particular mode of vibration, Eq. (55) can be reduced to the form

$$K^2 - i\gamma \frac{M}{(M^2-1)^{1/2}} \frac{M^2-2}{M^2-1} K - (1+i\eta_t)K_n^2 = 0 \tag{59}$$

where K and K_n are reduced frequency defined by

$$K \equiv \omega L / a_\infty, \qquad K_n \equiv \omega_n L / a_\infty$$

and $\gamma = \rho_\infty L / \rho h$ is the nondimensional mass ratio. The roots of characteristic equation (59) determine the stability of the beam plate. Typical results are shown in Fig. 5, where M_F is the Mach number at which flutter instability starts to occur, and is determined by the equation

$$\gamma M_F (M_F^2 - 2) + 2K_n (M_F^2 - 1)(M_F^2 - 1)^{1/2} = 0 \tag{60}$$

When $\eta_t = 0$, the stability boundary is at $M_S = \sqrt{2}$. In this case, the beam plate is unstable when $M<\sqrt{2}$ due to the negative aerodynamic damping and is stable when $M>\sqrt{2}$ due to the positive damping. In the case of $\eta_t \neq 0$, the stability boundary is shifted to the lower Mach number, $M_S^* \cong M_S - (K_n/4\gamma)\eta_t$, to extend the stable range of Mach number by $\mathcal{O}(\eta_t)$. This result implies that thermal damping improves the stability of beam plates. Figure 5 shows that divergence instability may exist at $1<M<M_F$ depending on

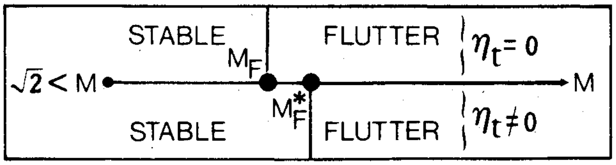

Fig. 6 Instability of coupled multiple modes ($N = 2$).

γ and K_n, when $\eta_t = 0$. However it also shows that divergence instability and thermal damping do not coexist.

Case II: Multiple Modes ($N = 2$)

Flutter may also occur as a result of coupling between vibration modes at high supersonic Mach numbers ($M > \sqrt{2}$). When we consider two particular modes at $M > \sqrt{2}$, Eq. (55) can be approximately written

$$K^4 - (1 + i\eta_t)(K_1^2 + K_2^2)K^2 + (1 + i\eta_t)^2 K_1^2 K_2^2 + \lambda^2 \Phi^2 = 0 \quad (61)$$

where

$$\Phi^2 = \left[\frac{M^4}{M^2 - 1}\right] \left\{ \left[\int_0^1 \frac{dW_1}{d\xi} W_2 d\xi \right]^2 \Big/ \int_0^1 W_1^2 d\xi \int_0^1 W_2^2 d\xi \right\} = \left[\frac{M^4}{M^2 - 1}\right] \phi^2$$

Flutter boundaries are given analytically as follows:

$$4\gamma^2 \phi^2 M_F^4 + (K_1^2 - K_2^2)^2 (M_F^2 - 1) = 0 \quad (62)$$

$$M_F^* = M_F + [(K_1^2 + K_2^2)^2 / 16 M_F^3 \gamma \phi^2] \eta_t^2 \quad (63)$$

Figure 6 shows the effects of thermal damping on the flutter of coupled multiple modes. These results again demonstrate that thermal damping is favorable for stability, by $\mathcal{O}(\eta_t^2)$.

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